Section 0.4 Mathematical Induction (Optional)

In Section 1 we introduced the ideas of elementary logic to provide a basis upon which to introduce techniques of proof. The discussion in Section 0.2 of techniques of proof can be succinctly summarized as follows:

From a stated hypothesis reason to a valid conclusion.

The term "reason" can be thought of as building a logic bridge. Our building blocks were other truths. The connection of one truth to the next formed our bridge from the hypothesis to the conclusion. This procedure is called **deductive reasoning** and is generally what is meant by a formal proof in mathematics. A new truth is deduced from other truths.

Another type of reasoning is known as **inductive reasoning**. Inductive reasoning involves collecting evidence from experiments or observations and using this information to formulate a general law or principle. Inductive reasoning attempts to go from the specific to the general, but even with large quantities of evidence the conclusion is not guaranteed. In general, mathematics rejects direct inductive reasoning. However, mathematics often uses an inductive process to formulate conjectures which are then subjected to rigorous deductive reasoning before they are accepted. Remember that a conjecture is a conclusion based on incomplete evidence—that is, a guess. Much can be gained from using experimental evidence to suggest conjectures and then applying deductive arguments to determine the truth or falsity of the conjecture. A number of important advances in science and engineering evolved in just this way.

The cycle of

experiment(s) — conjecture — check by deductive reasoning

is very important in mathematics. Here we introduce one method for performing the "deductive check" on certain conjectures that can be derived from experiments that involve using the natural numbers. (The natural numbers N are the set $\{1, 2, 3, \dots\}$.) The process we describe below is called **Mathematical Induction**, but it is really a particular deductive checking procedure *not* reasoning by induction.

Experiments are often performed in mathematics to determine patterns of behavior. Patterns of behavior are often reformulated into mathematical properties and mathematical theorems. Experiments in mathematics often take the form of looking at special cases and trying to see some common pattern in order to derive a conjecture. The special cases in the patterns we explore are obtained by using the first "few" natural numbers.

Example 1. Determine a conjecture for a formula to compute the sum of consecutive natural numbers.

Discussion: The experiments for the cases of 1, 2, 3, 4, and 5 consecutive natural numbers are shown next.

$$\begin{aligned}
1 &= 1 \\
1 + 2 &= 3 \\
1 + 2 + 3 &= 6 \\
1 + 2 + 3 + 4 &= 10 \\
1 + 2 + 3 + 4 + 5 &= 15
\end{aligned}$$

Looking for a pattern in the sums is not easy. Try to connect the sum to the largest integer used in the set of addends. (We do not always have to start looking for a pattern at the "beginning".) Looking at

$$1 + 2 + \boxed{3} \qquad = \boxed{6}$$
$$1 + 2 + \ 3 \ + \boxed{4} \ = 10$$

we see that $\boxed{3}\boxed{4} = 12 = 2\boxed{6}$. Looking at

$$1 + 2 + 3 + \boxed{4} \qquad = \boxed{10}$$
$$1 + 2 + 3 + \ 4 \ + \boxed{5} = \ 15$$

we see that $\boxed{4}\boxed{5} = 20 = 2\boxed{10}$. Checking the first two sums we see that the same pattern holds. For

$$\boxed{1} \qquad = \boxed{1}$$
$$1 \ + \boxed{2} \quad = \ 3$$

$\boxed{1}\boxed{2} = 2 = 2\boxed{1}$. Checking the second pair of sums we see that the same pattern holds. (Verify.) Next we adjoin another row to the experiment table above:

$$1 + 2 + 3 + 4 + 5 + 6 = 21$$

Checking the last pair of rows:

$$\boxed{5}\boxed{6} = 2\boxed{15}$$

so the pattern holds here also. To formulate a conjecture we proceed as follows. Let the largest natural number used in a sum of consecutive natural numbers be k. Then we have

$$\begin{aligned} 1 + 2 + \cdots + k - 1 + \boxed{k} \qquad\qquad &= S_k \\ 1 + 2 + \cdots + k - 1 + \ k \ + \boxed{k+1} &= S_{k+1} \end{aligned}$$

The pattern above suggests that

$$k(k+1) = 2S_k$$

or equivalently

$$S_k = \frac{k(k+1)}{2}$$

Using summation notation (see Section 1.2) we have

$$S_k = 1 + 2 + \cdots + k = \sum_{i=1}^{k} i$$

Hence our conjecture is

If we form the sum S_k of the first k consecutive natural numbers then $S_k = \sum_{i=1}^{k} = \dfrac{k(k+1)}{2}$.

We have only done the experiment and conjecture steps of our cycle. Before we can claim that the sum of the first k consecutive natural numbers is $k(k+1)/2$ we must use deductive reasoning. (See Example 3.) ■

Example 2. Determine a conjecture for a relationship between the expressions $(1+x)^k$ and $1 + kx$ where x is such that $1 + x > 0$ and k is any natural number.

Discussion: We experiment with the first few natural numbers k and various values of x.

Case $k=1$:	x any real number so that $x > -1$		
	$(1+x)^1 = 1+1x$		
Case $k=2$:	$x=-0.5$	$(1+(-0.5))^2 = 0.25$	$1+2(-0.5)=0$
	$x=0$	$(1+0)^2=1$	$1+2(0)=1$
	$x=1$	$(1+1)^2=4$	$1+2(1)=3$
	$x=15$	$(1+15)^2=256$	$1+2(15)=31$
	Summary: for the few cases above, $(1+x)^2 > 1+2x$		
Case $k=3$:	$x=-0.75$	$(1+(-0.75))^3 = 0.015625$	$1+3(-0.75) = -1.25$
	$x=-0.1$	$(1+(-0.1))^3 = 0.970299$	$1+3(-0.1)=0.7$
	$x=2$	$(1+2)^3=27$	$1+3(2)=7$
	$x=7$	$(1+7)^3=512$	$1+3(7)=22$
	$x=20$	$(1+20)^3=9261$	$1+3(20)=61$
	Summary: for the few cases above, $(1+x)^3 > 1+3x$		

This limited evidence suggests that we form the conjecture

If k is any natural number and $x > -1$, then $(1+x)^k \geq 1+kx$.

Before we can claim that this relationship is true we must use deductive reasoning to supply a proof. (See Example 4.) ∎

Note that the conjecture in both examples is stated as a conditional $p \implies q$. The technique of Mathematical Induction which we define next is a method of proof for the special kind of conditional statements which appear in Examples 1 and 2. The principle of **Mathematical Induction** is stated as follows:

> If S is a set of natural numbers with
>
> (a) $1 \in S$
>
> and
>
> (b) $n \in S \implies n+1 \in S$ for each natural number n,
>
> then $S = N$, the set of all natural numbers.

The set S is the set of natural numbers for which the conjecture $p \implies q$ is true. The method of mathematical induction says

first: show $p \implies q$ for $n=1$
next: assume $p \implies q$ for arbitrary natural number n and *prove* that $p \implies q$ is true for natural number $n+1$.

If both steps are successful the principle of mathematical induction guarantees that $p \implies q$ is true for *every* natural number.

Warning: The proof for part (b) depends on the contents of the implication $p \implies q$ and many times requires ingenuity.

To illustrate the principle of mathematical induction we prove the conjectures developed in Examples 1 and 2. (Terminology: The name mathematical induction is often shortened to just **induction**.)

Example 3. Apply induction to prove the conjecture developed in Example 1. Let S be the set of all natural numbers k for which

$$\sum_{i=1}^{k} = \frac{k(k+1)}{2}$$

From the experiments in Example 1, we have 1, 2, 3, 4, 5 $\in S$. The principle of induction says assume that $n \in S$; that is,

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{1}$$

Then we must verify that $n+1 \in S$; that is, we must show

$$\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + n + 1 = \frac{(n+1)(n+2)}{2} \tag{2}$$

(Note that n in the right-hand side of (1) was replaced by $n+1$ to obtain the right-hand side of (2).) In order to verify that (2) is true we use (1) which is assumed to be true. (The expression in (1) is called the **induction hypothesis** for this conjecture.) Starting with the expression

$$\sum_{i=1}^{n+1} i$$

we must show algebraically that we can produce the formula

$$\frac{(n+1)(n+2)}{2}.$$

We have

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) && \{\text{by properties of sums}\}\\
&= \frac{n(n+1)}{2} + (n+1) && \{\text{by (1)}\}\\
&= \frac{n(n+1) + 2(n+1)}{2} && \{\text{by algebra}\}\\
&= \frac{(n+1)(n+2)}{2} && \{\text{by factoring}\}
\end{aligned}$$

Hence we have shown that if $n \in S$ then $n+1 \in S$. Thus the principle of induction implies that $S = N$; that is, the formula

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

is valid for all natural numbers. ■

The deductive reasoning in the principle of mathematical induction is the proof *we must supply* to show part

$$\text{(b): } n \in S \Longrightarrow n+1 \in S.$$

This step is itself a conditional statement that must be proven using its hypothesis, $n \in S$, and appropriate building blocks. The inductive hypothesis, $n \in S$, does *not* mean we are assuming what we want to prove. We are not assuming what we want to prove because we must supply a proof that $n+1 \in S$. Thus to prove (b) we must adhere to the rules about proving conditionals which are stated in Table 8 in Section 0.1. From Table 8 we see that (b) is false only if n is in S but $n+1$ is not. Hence if we can show that whenever $n \in S$ that it must follow that $n+1 \in S$, then the conditional (b) is always true. In summary, *assuming $n \in S$ does not assume what we must prove*, namely that $n+1 \in S$.

Example 4. Apply induction to prove the conjecture developed in Example 2. Let S be the set of all natural numbers k such that

$$(1+x)^k \geq 1+kx, \quad \text{whenever } 1+x>0$$

From the experiments in Example 2 we have that $1 \in S$ and we suspect that 2 and 3 belong to S. It is only a suspicion since we have not verified the conjecture for all $x > -1$ when $k = 2$ or 3. We next verify the conditional in (b). Assume that $n \in S$; that is,

$$(1+x)^n \geq 1+nx \quad \text{for all } x > -1$$

We must prove deductively that $n+1 \in S$: that is, prove that

$$(1+x)^{n+1} \geq 1+(n+1)x \quad \text{for all } x > -1$$

We proceed as follows

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) && \{\text{by algebra}\} \\
&\geq (1+nx)(1+x) && \{\text{by the inductive hypothesis}\} \\
&= 1+nx+x+nx^2 && \{\text{by algebra}\} \\
&= 1+(n+1)x+nx^2 && \{\text{by algebra}\} \\
&\geq 1+(n+1)x && \{\text{since } nx^2>0\}
\end{aligned}$$

Hence by the principle of induction

$$(1+x)^k \geq 1+kx$$

for all $x > -1$ and for all natural numbers k. ■

Most cases in which you may need to use induction in linear algebra are already phrased as prove: $p \implies q$. The experiments and conjecture formulation stages of the process we have described have been done. You must recognize that the special structure of inductive proofs is present. That is, the natural numbers play a role in $p \implies q$. The building blocks in proving part (b) of the principle of induction will most likely be facts about matrices or other linear algebra concepts rather than ordinary algebra facts as in Examples 3 and 4. Proof by induction appears in only a few places in this manual but it is an important mathematical technique. Places where induction can be used are listed next:

Section 1.4 Exercise 7;
Supplementary Exercises for Chapter 1 Exercises 1(d) and 15;
Section 4.4 Exercise 23;
Supplementary Exercises for Chapter 5 Exercise 3.

For further reading on induction see the following sources

[1] H. Burrows, et al, Mathematical Induction, FIAM Module, COMAP, 1989.

[2] S. Lay, **Analysis, An Introduction to Proof**, Prentice Hall, 1986.

[3] L. Swanson and R. Hansen, Mathematical Induction or "What Good is All This Stuff if We Are Going to Assume It's True Anyway?", Two Year College Mathematics Journal, v.12, 1981, pp. 8–12.

[4] B. Youse, **Mathematical Induction**, Prentice Hall, 1964.