# Brief overview of the notion of homogeneous coordinates.

The translation of a point, vector, or object defined by a set of points in the plane is performed by adding the same quantity $\Delta x$ to each x-coordinate and the same quantity $\Delta y$ to each y-coordinate. (We emphasize that $\Delta x$ and $\Delta y$ are not required to be equal in magnitude.) We illustrate this in Figure 1 for a point (x,y) in $R^2$, where the coordinates of the translated point are $(x^*, y^*) = (x+\Delta x, y+\Delta y)$.

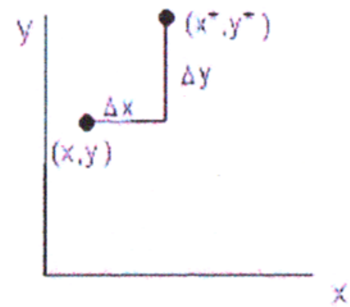


Figure 1.

It is easy to show that the operation of translation given by $\mathbf{T}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + \Delta x \\ y + \Delta y \end{bmatrix}$ is not a linear transformation. (Verify.) Thus we can not perform a translation in $R^2$ using multiplication by a 2 by 2 matrix. In order to have rotations, contractions and expansions, shears, and projections "play together nicely" with translations (that is, each can be performed by matrix multiplication) we change the space in which we work. To employ matrix multiplication to perform translations we adjoin another component to vectors and border matrices (See Figure 2.) with another row and column. This change is said to use **homogeneous coordinates**. To use homogeneous coordinates we make the following identifications.

> A vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $R^2$ is identified with the vector $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ in $R^3$.
>
> The first two coordinates are the same and the third coordinate is 1.

Each of the matrices **M** associated with plane linear transformations is now identified with a 3 × 3 matrix of the form

$$\begin{bmatrix} \mathbf{A} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Figure 2.

For example when using homogeneous coordinates for a reflection about the y-axis the corresponding matrix is the 3 × 3 matrix $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Also when using homogeneous coordinates for a rotation by an angle $\theta$ the corresponding matrix is the 3 × 3 matrix $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. A translation can be performed by matrix multiplication on data expressed in homogeneous coordinates using the 3 × 3 matrix $\begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix}$. We have $\begin{bmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+\Delta x \\ y+\Delta y \\ 1 \end{bmatrix}$.